SNSB
Summer Term 2013
Ergodic Theory and Additive Combinatorics
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Seminar 1

(S1.1) Define an equivalence relation on $\mathbb{R}$ by

$$x \sim y \text{ if and only if } x - y \in \mathbb{Z}, \tag{2.13}$$

let $\mathbb{R}/\mathbb{Z}$ be the set of equivalence classes $[x]$, and $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the natural projection. Endow $\mathbb{R}/\mathbb{Z}$ with the quotient topology and for every $\alpha \in [0, 1)$ define

$$T_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad T_\alpha([x]) = [x + \alpha].$$

Prove that $(\mathbb{R}/\mathbb{Z}, T_\alpha)$ is a TDS isomorphic with $(\mathbb{S}^1, R_a)$, where $\alpha \in [0, 1)$ and $a = e^{2\pi i\alpha}$.

Let $\mathcal{F}$ be a collection of blocks over W, which we will think of as being the **forbidden blocks**. For any such $\mathcal{F}$, define $X_\mathcal{F}$ to be the set of sequences which do not contain any block in $\mathcal{F}$.

Definition 2.3.8. *A* **shift space** *(or simply* **shift***) is a subset X of a full shift $W^\mathbb{Z}$ such that $X = X_\mathcal{F}$ for some collection $\mathcal{F}$ of forbidden blocks over W.*

Note that the empty space is a shift space, since putting $\mathcal{F} = W^\mathbb{Z}$ rules out every point. Furthermore, the full shift $W^\mathbb{Z}$ is a shift space; we can simply take $\mathcal{F} = \emptyset$, reflecting the fact that there are no constraints, so that $W^\mathbb{Z} = X_\mathcal{F}$.

The collection $\mathcal{F}$ may be finite or infinite. In any case it is at most countable since its elements can be arranged in a list (just write down its blocks of length 1 first, then those of length 2, and so on).

Definition 2.3.9. *Let X be a subset of the full shift $W^\mathbb{Z}$, and let $\mathcal{B}_n(X)$ denote the set of all n-blocks that occur in points of X. The* **language of** *X is the collection*

$$\mathcal{B}(X) = \bigcup_{n \geq 0} \mathcal{B}_n(X). \tag{2.14}$$

For a block $u \in \mathcal{B}(X)$, we say also that u **occurs in** X or x **appears in** X or x **is allowed in** X.

(S1.2) Let $X \subseteq W^{\mathbb{Z}}$ be a nonempty subset of $W^{\mathbb{Z}}$.

(i) $X \subseteq X_{\mathcal{B}(X)^c}$.

(ii) If X is a shift space, then $X = X_{\mathcal{B}(X)^c}$. Thus, the language of a shift space determines the shift space.

(S1.3) Let $X \subseteq W^{\mathbb{Z}}$ be a nonempty subset of $W^{\mathbb{Z}}$. The following are equivalent

(i) X is a shift space.

(ii) For every $\mathbf{x} \in W^{\mathbb{Z}}$, if $\mathbf{x}_{[i,j]} \in \mathcal{B}(X)$ for all $i \geq j \in \mathbb{Z}$, then $\mathbf{x} \in X$.

(iii) X is a closed strongly T-invariant subset of $W^{\mathbb{Z}}$.

(S1.4) Determine whether the following sets are shift spaces or not:

(i) X is the set of all binary sequences with no two 1's next to each other.

(ii) X is the set of all binary sequences so that between any two 1's there are an even number of 0's.

(iii) X is the set of points each of which contains exactly one symbol 1 and the rest 0's.